Jacobi fields

$f(t,s)$ - a 1-parameter smooth family of geodesics in (M,g)

given $s=s_0$ $\quad t \to \gamma_{s_0}(t) = f(t,s_0)$ is an affinely parametrized geodesic in M

$$\Sigma = \{ M \ni p : \ p = f(t,s),\ 1 \le t \le 1,\ -\varepsilon \le s \le \varepsilon \}$$

Two vector fields: $u = \frac{\partial}{\partial t}$ and $J = \frac{\partial}{\partial s}$ on Σ!

We have

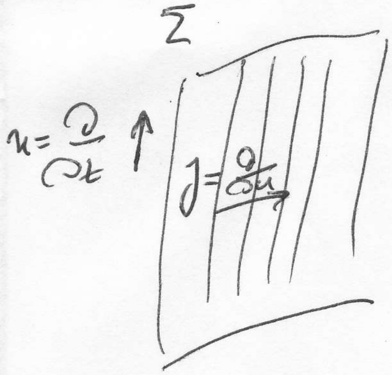
$$\nabla_u u = 0 \quad \text{since } \gamma_{s_0}(t) \text{ is geodesic for each } s=s_0$$

$$0 = \nabla_J \nabla_u u \underset{\uparrow}{=} \nabla_u \nabla_J u - R(u,J)u =$$

$$\nabla_u \nabla_J u - \nabla_J \nabla_u u - \cancel{\nabla_{[u,J]}} u = R(u,J)u$$

$$\underset{\uparrow}{=} \nabla_u^2 J - R(u,J)u$$

$$\nabla_J u - \nabla_u J - \cancel{[J,u]} = 0$$



$$\boxed{\nabla_u^2 J - R(u,J)u = 0}$$ Jacobi equation (geodesic deviation equation)

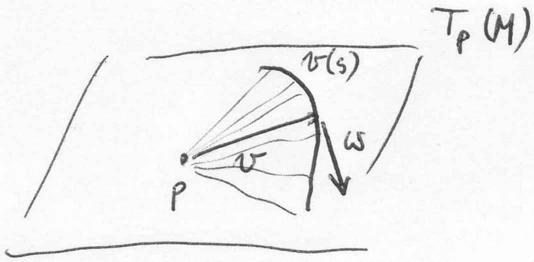
Def

A vector field J along a geodesic $\gamma : [0,a] \to M$ which satisfies

$$\frac{D^2 J}{dt^2} - R(\gamma', J)\gamma' = 0$$

is called a Jacobi field.

Let $v \in T_p M$ be such that $\exp_p(v)$ is defined.



Let $w \in T_v(T_p(M))$

Consider a curve

$s \mapsto v(s) \in T_p(M)$ s.t.

$$\begin{cases} v(0) = v \\ \left.\dfrac{dv}{ds}\right|_{s=0} = w \end{cases} \qquad -\varepsilon < s \le \varepsilon$$

and a surface

$$\Sigma = \{ M \ni f(t,s) = \exp_p(t\, v(s)) \}_{0 \le t \le 1}$$

We are in the previous situation since $p(t, s_0)$ is a geodesic.

$$(\exp_p)_{*v}(w) = \left.\frac{d}{ds}\right|_{s=0} \exp_p(v(s)) = \frac{\partial f}{\partial s}(1,0)$$

$$\parallel$$

$$(d\exp_p)_v(w)$$

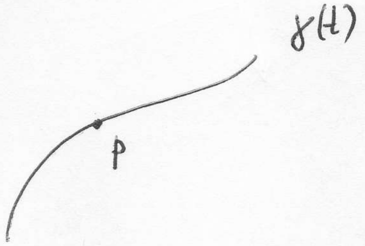
More generally:

$$(d\exp_p)_{tv}(tw) = \left.\frac{d}{ds}\right|_{s=0} \exp_p(t\, v(s)) = \frac{\partial f}{\partial s}(t,0)$$

$$\Rightarrow \boxed{\frac{D^2}{dt^2}\frac{\partial f}{\partial s} - R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t} = 0}$$

Local expression

$\gamma(t)$ – geodesic s.t. $\gamma(0)=p$



Choose an orthonormal frame $X_\mu(0)$ at p and propagate it parallelly along $\gamma(t)$.

$\Rightarrow$ frame $X_\mu(t)$ along $\gamma(t)$.

If $J(t)$ is a Jacobi field then

$$J(t) = J^\mu(t)\, X_\mu(t) \quad \text{and}$$

$$\frac{D^2 J}{dt^2} = \ddot{J}^\mu(t)\, X_\mu(t) \qquad \text{since } \frac{DX_\mu}{dt} = 0.$$

$$\gamma'(t) = \frac{dx^\mu}{dt} X_\mu(t)$$

$$R(\gamma'(t), J(t))\gamma'(t) = R^\mu{}_{\nu\rho\sigma}\, \dot{x}^\rho J^\sigma \dot{x}^\nu X_\mu =$$

$$= a^\mu{}_\sigma(t)\, J^\sigma(t)\, X_\mu(t)$$

where $a^\mu{}_\sigma(t) = R^\mu{}_{\nu\rho\sigma}(t)\,\dot{x}^\nu(t)\,\dot{x}^\rho(t)$

$$\Rightarrow \qquad \boxed{\ddot{J}^\mu(t) = a^\mu{}_\sigma(t)\, J^\sigma(t)}$$

is a linear second order system for the unknowns $J^\mu(t)$.

$\Rightarrow$ $(J^\mu(0), \dot{J}^\mu(0))$ initial conditions

$\Rightarrow$ <u>2n linearly independent solutions</u>! of class C^∞ on $[0,a]$.

Note that $J_1 = \gamma'(t)$ and $J_2 = t\gamma'(t)$ are Jacobi fields!

$$\left\{ \begin{array}{ll} J_1 \text{ is nonvanishing and } & \frac{DJ_1}{dt} \equiv 0 \\ J_2(0) = 0 & \frac{DJ_2}{dt}(0) = \gamma'(0) \end{array} \right. \Rightarrow$$ J_1 and J_2 are linearly independent.

It is sufficient to look for $2n-2$ linearly independent solutions which are orthogonal to $\gamma'(t)$.

<u>Example</u> Jacobi fields on manifolds of constant curvature.

We can always use an affine parameter such that $|\gamma'(t)| = 1$

arc length

Take J s.t. $g(\gamma'(t), J(t)) = 0$, $J(t) \neq 0$.

$$R(\gamma'(t), J(t))\gamma'(t) = R^{\mu}{}_{\nu\rho\sigma}(t)\dot{x}^{\nu}(t)\dot{x}^{\rho}(t)J^{\sigma}(t)X_{\mu}(t)$$

$$R^{\mu}{}_{\nu\rho\sigma}(t) = K\left(\delta^{\mu}_{\rho}g_{\nu\sigma} - \delta^{\mu}_{\sigma}g_{\nu\rho}\right)$$

$$R(\gamma'(t), J(t))\gamma'(t) = K\left(\dot{x}^{\mu}g(\gamma', J) - J^{\mu}|\gamma'|^2\right)X_{\mu} =$$

$$= -K \cdot J$$

$\Rightarrow$ Jacobi equation:

$$\boxed{\frac{D^2 J}{dt^2} + KJ = 0}$$ $K = \text{const.}$

Let $\omega(t)$ be parallel along $\gamma(t)$ and such that

$$g(\omega(t), \gamma'(t)) = 0, \qquad g(\omega(t), \omega(t)) = 1$$

$J(t) = j(t)\,\omega(t)$ and

$$\frac{d^2 j}{dt^2} + K j = 0 \quad \Rightarrow$$

$$J(t) = \begin{cases} \dfrac{\sin t\sqrt{K}}{\sqrt{K}}\,\omega(t) & \text{if } K > 0 \\ t\,\omega(t) & \text{if } K = 0 \\ \dfrac{\sinh t\sqrt{-K}}{\sqrt{-K}}\,\omega(t) & \text{if } K < 0. \end{cases}$$

is a solution for the Jacobi equation satisfying

$$J(0) = 0, \quad J'(0) = \omega'(0).$$

Note that if $K > 0$ there exists $t_0 = \frac{\pi}{\sqrt{K}}$ s.t.

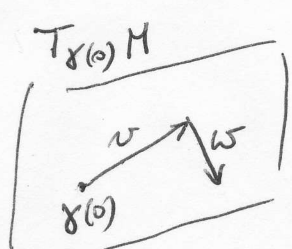
$$J(t_0) = J(0) = 0,$$

Such $t_0 \neq 0$ does not exist if $K \leq 0$!

Proposition

Let $\gamma : [0, a] \to M$ be a geodesic and let J be a Jacobi field along γ with $J(0) = 0$. Let $\omega = \frac{DJ}{dt}\big|_{t=0}$ and $v = \frac{d\gamma}{dt}\big|_{t=0}$

Consider a curve $v(s)$ in $T_{\gamma(0)}$ s.t.



$$v(0) = a v$$

$$\frac{dv}{ds}\Big|_{s=0} = w$$

and 2-dimensional surface in M given by

$$f(t,s) = \exp_{\gamma(0)}\left(\frac{t}{a} v(s)\right) \Rightarrow J(t) = \frac{\partial f}{\partial s}(t, 0)$$

To prove it is enough to check the initial conditions.

γ - a geodesic s.t.

$\gamma(0)=p$

$\gamma'(0)=v$

Let $w \in T_v(T_pM)$ with $|w|=1$ and Jacobi field

$J(t)=(d\exp_{\gamma(0)})_{tv}(tw)$

Taylor expansion

$\Rightarrow |J(t)|^2 = t^2 + \frac{1}{3} g(R(v,w)v,w)t^4 + \theta(t)$

$\frac{\theta(t)}{t^4} \xrightarrow[t\to 0]{} 0.$

<u>Proof</u>

$J(0)=0$

$\frac{DJ}{dt}(0)=w$

$\Rightarrow \quad |J(0)|^2=0$

$(|J(0)|^2)' = g(J(t),J(t))'\big|_{t=0} = 2\,g(J(0),J'(0))=0$

$(|J(0)|^2)'' = 2\,g(J'(0),J'(0)) + 2\,g(J(0),J''(0)) = 2$

$(|J(0)|^2)''' = 6\,g(J'(0),J''(0)) + 2g(J(0),J'''(0)) = 0$

$J''(0) = R(\gamma',J)\gamma'(0) = 0$

$(\nabla^2_{\gamma'} J)(0)$

Jacobi equation

$J'''(0) = (\nabla^3_{\gamma'} J)(0) = \nabla_{\gamma'}(R(\gamma',J)\gamma')(0) = (R(\gamma',J')\gamma')(0)$

since all the other terms from differentiation are zero since $J(0)=0$.

$(J(0)^2)'''' = 8\,g(J'(0),J'''(0)) =$

$= 8\,g(R(v,w)v,w)$

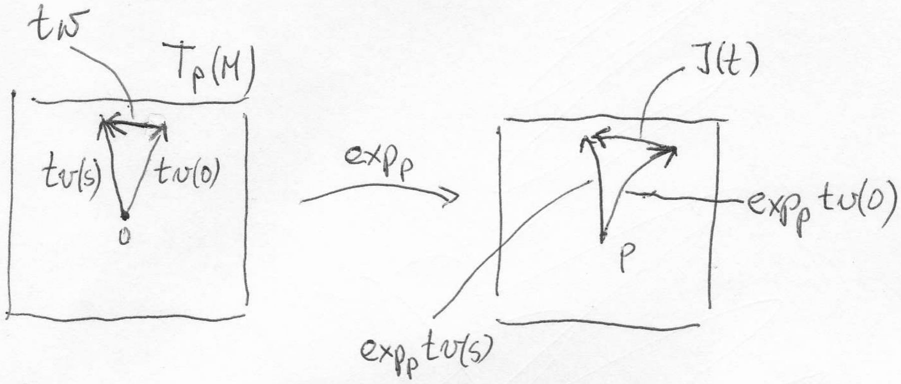
Taylor expansion $\Longrightarrow$ $|J(t)|^2 = t^2 + \frac{8}{24} g(R(v,w)v,w)t^4 + \theta(t^5)$

If γ is parametrized by arc length we have $|\omega|=1$
and g

$$\Rightarrow \quad |J(t)|^2 = t^2 - \frac{1}{3}K(p,\sigma)t^4 + O(t^5)$$

where $K(p,\sigma)$ is a sectional curvature at p
w.r.t. the plane generated by v and ω.

$$\Rightarrow \quad |J(t)| = t - \frac{1}{6}K(p,\sigma)t^3 + O(t^4)$$



$|t\omega| = t$ $\qquad$ $|J(t)| = t - \frac{1}{6}K(p,\sigma)t^3$

so if $K(p,\sigma) > 0$ geodesics are <u>converging</u> quicker
than in $T_p(M)$

$K(p,\sigma) < 0$ — " — <u>diverging</u> — " —
— " —